OPTIMUM RELATIONSHIPS IN A PLANE MULTILAYER WALL WITH INTERNAL HEAT RELEASE

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Inzhenerno-Fizicheskii Zhurnal, Vol. 15, No. 2, pp. 341–345, 1968

UDC 536.21

We consider the possibility of reducing the heat flows through a flat multilayer wall with internal heat release.

The existence of internal heat sources in thermal insulation leads to a large variety of problems which arise in calculations dealing with specific corresponding situations. With constant intensities of heat release and of the physical properties for each of the layers – only the thicknesses of the layers changing – we can find the relationship between the thickness of the layers at which the heat flow to the medium with the lower temperature is at a minimum, or we can find the relationship at which the flow of heat to the medium with the lower temperature is at a minimum while the heat flow from the heated medium to the wall is equal to zero. These variations on the problem arise in the design of nuclear reactors.

Let us deal initially with the first case.

In reference [1] we derived the relationships which determine the steady flow of heat through a multilayer wall with heat release.

Applying the formula for a flat wall to the calculation of the heat flow to a medium with a low temperature, we find that the magnitude of the flow

$$q_x = \Bigg[T_{11} - T_{12} + \frac{1}{2}\sum_{i=1}^{n} q_{vi}\frac{\delta_i^2}{\lambda_i} + \sum_{k=1}^{n-1}\frac{\delta_k}{\lambda_k}\sum_{i=k+1}^{n} q_{vi}\,\delta_i + \frac{1}{\alpha_1}\sum_{i=1}^{n} q_{vi}\,\delta_i\Bigg]\Bigg[\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \sum_{i=1}^{n}\frac{\delta_i}{\lambda_i}\Bigg]^{-1}. \quad (1)$$

Here the layers are numbered from the hot medium to the cold.

Let us isolate a layer with the number l and let us consider the magnitude of the heat flow as a function of the Layer's thickness.

If we employ the notation

$$\bar{\delta}_i = \frac{\delta_i}{\lambda_i}, \quad \Delta T = T_{11} - T_{12},$$

$$E_l = \Delta T + \frac{1}{2}\sum_{\substack{i=1\\ i\neq l}}^{n} q_{vi}\lambda_i\bar{\delta}_i^2 + \sum_{\substack{k=1\\ k\neq l}}^{n-1}\bar{\delta}_k\sum_{\substack{i=k+1\\ i\neq l}}^{n} q_{vi}\lambda_i\bar{\delta}_i + \frac{1}{\alpha_1}\sum_{\substack{i=1\\ i\neq l}}^{n} q_{vi}\lambda_i\bar{\delta}_i,$$

$$F_l = q_{vl}\lambda_l\left(\sum_{i=1}^{l-1}\bar{\delta}_i + \frac{1}{\alpha_1}\right) + \sum_{i=l+1}^{n} q_{vi}\lambda_i\bar{\delta}_i, \quad (2)$$

$$G_l = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \sum_{\substack{i=1\\ i\neq l}}^{n}\bar{\delta}_i,$$

we have

$$q_x = \frac{\frac{1}{2}q_{vl}\lambda_l\bar{\delta}_l^2 + F_l\bar{\delta}_l + E_l}{G_l + \bar{\delta}_l}. \quad (3)$$

From (3) we find the optimum thickness of this layer:

$$\bar{\delta}_{l\,\mathrm{opt}} = -G_l + \sqrt{G_l^2 + \frac{2}{q_{vl}\lambda_l}(E_l - F_l G_l)}. \quad (4)$$

This value corresponds to a minimum for q_x. Indeed, with a reduction in the thickness of the wall we find a reduced thermal resistance on the part of the wall, while with an increase in the wall thickness there is a corresponding rise in the quantity of heat generated within the wall.

When there are only a few layers, the formulas can be derived in a more convenient form.

The heat is transported through a single-layer wall. In this event

$$G = \frac{1}{\alpha_1} + \frac{1}{\alpha_2},$$

$$E = \Delta T = T_{11} - T_{12}, \quad F = \frac{q_v\lambda}{\alpha_1}. \quad (5)$$

Having substituted (5) into (4), we find that the wall thickness at which the heat flow to the medium with the lower temperature is at a minimum is given by

$$\delta_{\mathrm{opt}} = \lambda\left(-\frac{1}{\alpha_1} - \frac{1}{\alpha_2} + \sqrt{\frac{1}{\alpha_2^2} - \frac{1}{\alpha_1^2} + \frac{2\Delta T}{q_v\lambda}}\right), \quad (6)$$

while the magnitude of the minimum flow

$$q_{x\min} = q_v\lambda\left(-\frac{1}{\alpha_2} + \sqrt{\frac{1}{\alpha_2^2} - \frac{1}{\alpha_1^2} + \frac{2\Delta T}{q_v\lambda}}\right) = q_v\left(\delta_{\mathrm{opt}} + \frac{\lambda}{\alpha_1}\right). \quad (7)$$

It follows from (6) that δ_{opt} is positive if we satisfy the condition

$$\Delta T \Big/ \frac{q_v\lambda}{\alpha_1}\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right) > 1. \quad (8)$$

However, if this condition is not satisfied, the use of this thermal insulation leads to a rise in the heat flow to the medium that has not been heated as much.

Equations (6) and (7) can be written in criterial form

$$\mathrm{Bi}_{opt} = \frac{\delta_{opt}\alpha_1}{\lambda} = -1 - \frac{\alpha_1}{\alpha_2} + \sqrt{\left(\frac{\alpha_1}{\alpha_2}\right)^2 - 1 + \frac{2\Delta T \alpha_1^2}{q_v \lambda}}, \quad (6a)$$

$$\frac{q_{x\min}}{q_v \delta_{opt}} = 1 + \frac{1}{\mathrm{Bi}_{opt}}. \quad (7a)$$

The figure shows Bi_{opt} as a function of $K = \Delta T \alpha_1^2 / q_v \lambda$ and α_1/α_2.

The intersection of the α_1/α_2 = const lines with the axis of abscissas occurs in the region $K > 1$ in accordance with condition (8).

$$E_1 = \Delta T + \frac{1}{2} q_{v2}\lambda_2\bar{\delta}_2^2 + \frac{1}{\alpha_1} q_{v2}\lambda_2\bar{\delta}_2,$$

$$F_1 = \frac{q_{v1}\lambda_1}{\alpha_1} + q_{v2}\lambda_2\bar{\delta}_2,$$

$$G_1 = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \bar{\delta}_2. \quad (9)$$

For the second layer

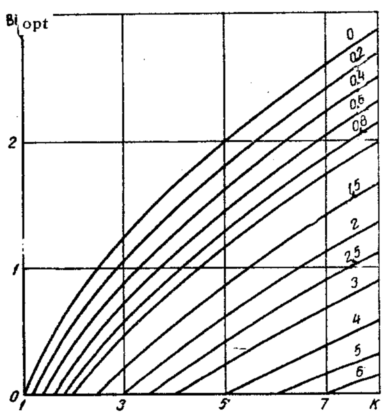
$$E_2 = \Delta T + \frac{1}{2} q_{v1}\lambda_1\bar{\delta}_1^2 + \frac{1}{\alpha_1} q_{v1}\lambda_1\bar{\delta}_1,$$

$$F_2 = q_{v2}\lambda_2\bar{\delta}_1 + \frac{q_{v2}\lambda_2}{\alpha_1},$$

$$G_2 = \frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \bar{\delta}_1. \quad (10)$$

Substuting (9) and (10) into (4), we find the optimization relationships:

$$\mathrm{Bi}_{1\,opt} = -1 - \frac{\alpha_1}{\alpha_2} - \mathrm{Bi}_2 + \left[\left(\frac{\alpha_1}{\alpha_2}\right)^2 - 1 + (1-\beta) \times \left(\mathrm{Bi}_2 + 2\frac{\alpha_1}{\alpha_2}\right)\mathrm{Bi}_2 + 2K_1\right]^{\frac{1}{2}}, \quad (11)$$



Graph for calculation of optimum thickness of single-layers heat insulation with inner heat generation (numbers near curves are values of α_1/α_2).

$$\mathrm{Bi}_{2\,opt} = -1 - \frac{\alpha_1}{\alpha_2} - \mathrm{Bi}_1 + \left[\left(\frac{\alpha_1}{\alpha_2}\right)^2 - 1 + \left(\frac{1}{\beta} - 1\right) \times (\mathrm{Bi}_1 + 2)\,\mathrm{Bi}_1 + 2K_2\right]^{\frac{1}{2}}, \quad (12)$$

where $\beta = q_{v2}\gamma_2 / q_{v1}\gamma_1$ and the notation introduced into (6a)

$$\mathrm{Bi}_{1,2} = \frac{\delta_{1,2}\alpha_1}{\lambda_{1,2}}, \quad K_{1,2} = \frac{\Delta T \alpha_1^2}{q_{v1,2}\lambda_{1,2}}.$$

These equations, in which the thickness of one of the layers—the main layer from the standpoint of structural considerations — is constant, enables us to determine the physical characteristics and the thickness for the main layer at which the use of additional insulating layers would be effective for the reduction of the heat flow through the wall.

The value of $\delta_{1\,opt}$ will be positive if $\bar{\delta}$ will vary in the limits

$$0 \leq \bar{\delta}_2 < \delta_{2\,\lim} = -\frac{1}{\alpha_2} - \frac{1}{\alpha_1\beta} + \left(\frac{1}{\alpha_2^2} - \frac{1-2\beta}{\alpha_1^2\beta^2} + \frac{2\Delta T}{q_{v2}\lambda_2}\right)^{\frac{1}{2}}. \quad (13)$$

These relationships are no simpler than the initial equation, and it is therefore better to use Eqs. (11) and (12).

The simultaneous solution of Eqs. (11) and (12) yields

$$\bar{\delta}_1 = -\frac{1}{\alpha_1},$$

$$\bar{\delta}_2 = -\frac{1}{\alpha_2} + \left(\frac{1}{\alpha_2^2} - \frac{1}{\beta\alpha_1^2} + \frac{2\Delta T}{q_{v2}\lambda_2}\right)^{\frac{1}{2}}.$$

The resulting nonpositive solution shows that the use of a single-layer wall with the smallest magnitude for $q_v\lambda$ is more effective than a multilayer wall. Thus, in structures with internal heat release the selection of a material exclusively on the basis of the thermal conductivity is without sufficient justification.

Let us now consider the case in which the insulation is chosen to ensure that the flow of heat from the hot medium to the wall is equal to zero. This imposes the additional condition

$$\Delta T - \frac{1}{2}\sum_{i=1}^{n} q_{vi}\lambda_i\bar{\delta}_i^2 - \sum_{k=2}^{n}\bar{\delta}_k \sum_{i=1}^{k-1} q_{vi}\lambda_i\bar{\delta}_i - \frac{1}{\alpha_2}\sum_{i=1}^{n} q_{vi}\lambda_i\bar{\delta}_i = 0. \quad (14)$$

For a single-layer wall we have

$$\mathrm{Bi}_{opt} = \frac{\delta_{opt}\alpha_2}{\lambda} = -1 + \sqrt{1 + 2\frac{\Delta T \alpha_2^2}{q_v\lambda}},$$

$$q_x = q_v\delta_{opt}, \quad (15)$$

while for a two-layer wall condition (14) makes possible the following expressions:

$$\mathrm{Bi}_{1\,opt} = -1 - \mathrm{Bi}_2 + [1 + (1-\beta)(\mathrm{Bi}_2 + 2)\mathrm{Bi}_2 + 2K_1]^{\frac{1}{2}}, \quad (16)$$

$$\mathrm{Bi}_{2\,\mathrm{opt}} = -1 - \frac{\mathrm{Bi}_1}{\beta} + \left[1 + \frac{1}{\beta}\left(\frac{1}{\beta} - 1\right)\mathrm{Bi}_1^2 + 2K_2\right]^{\frac{1}{2}}. \qquad (17)$$

The heat-transfer coefficient α_2 is contained in the complexes of formulas (15), (16), and (17).

In analogy with the previous case, it may develop that even here the use of a single-layer insulation with a smaller value for $q_\nu \lambda$ is the more effective.

These formulas can be used for calculations involving multilayer walls with internal heat release when the use of several layers is dictated by conditions of material compatibility, strength considerations, and the light.

NOTATION

q_x is the heat-flux density; q_ν is the power of internal heat sources; δ is the plate thickness; γ is the thermal conductivity; T is the temperature of medium; α is the heat transfer coefficient.

REFERENCES

1. M. S. Trakhtengerts, IFZh, no. 2, 1964
2. P. Schneider, Conduction Heat Transfer [Russian translation], IL, 1960.

14 December 1967 Power Engineering Institute, Moscow